On the Prior and Posterior Distributions Used in Graphical Modelling

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Background and Notation

The Problem

A large part of the literature on the analysis of graphical models focuses on the study of the parameters of local probability distributions (such as conditional probabilities or partial correlations). However:

- Comparing models learned with different algorithms is difficult, because they maximise different scores, use different estimators for the parameters, work under different sets of hypotheses, etc.
- Unless the true global probability distribution is known it is difficult to assess the quality of the estimated models.
- The few available measures of structural difference are completely descriptive in nature (e.g. Hamming distance [6] or SHD [13]), and are difficult to interpret.
- When learning causal graphical models often the focus is not on the parameters but in the presence of particular patterns of edges in the graph (e.g. [11]).

Aims of the Investigation

Focusing on graph structures makes sidesteps some of these problems, opens new ones and acknowledges the focus on graphs in part of causal modelling literature [12].

0. We need to know more about the properties of priors $\mathrm{P}(\mathcal{G})$ and posteriors $\mathrm{P}(\mathcal{G} \mid \mathcal{D})$ distributions over the space of graphs, preferably as a function of arc and edge sets, say $\mathrm{P}(\mathcal{G}(\mathcal{E}))$ and $\mathrm{P}(\mathcal{G}(\mathcal{E}) \mid \mathcal{D})$.

And then:

1. It would be good to have a measure(s) of spread for $\mathcal{G}$, to assess the noisiness of $\mathrm{P}(\mathcal{G}(\mathcal{E}) \mid \mathcal{D})$ and the informativeness of $\mathrm{P}(\mathcal{G}(\mathcal{E}))$.
2. Using such a measure(s), it would be interesting to study the convergence speed of structure learning algorithms and the influence of their tuning parameters.
3. It would also be interesting to investigate how to use higher order moments of $\mathrm{P}(\mathcal{G}(\mathcal{E}))$ to define new priors.

Notation

Graphical models are defined by:

- a network structure, either an undirected graph $\mathcal{G} = (\mathbf{V}, E)$ (Markov networks [2, 14]) or a directed acyclic graph $\mathcal{G} = (\mathbf{V}, A)$ (Bayesian networks [7, 8]). E is the edge set and A is the arc set. Each node $v \in \mathbf{V}$ corresponds to a random variable $X_i \in \mathbf{X}$;
- a global probability distribution over $\mathbf{X}$ with parameter set Θ, which can be factorised into a small set of local probability distributions according to the topology of the graph.

In addition, we denote $\mathcal{E} = \{(v_i, v_j), i \neq j\}$ the set of all possible edges or arcs of $\mathcal{G}$. Clearly, $|\mathcal{E}| = \mathcal{O}(|\mathbf{V}|^2)$ while the space of the graphs is at least $\mathcal{O}(2^{|\mathbf{V}|^2})$ so it is much bigger.

Modelling Graphs through Edges and Arcs

Edges and Univariate Bernoulli Random Variables

Each edge e_{ij} in an undirected graph $\mathcal{G} = (\mathbf{V}, E)$ has only two possible states,

$$e_{ij} = \begin{cases} 1 & \text{if } e_i \in E \\ 0 & \text{otherwise} \end{cases}.$$

Therefore it can be modelled as a Bernoulli random variable E_{ij},

$$e_{ij} \sim E_{ij} = \begin{cases} 1 & e_{ij} \in E \text{ with probability } p_{ij} \\ 0 & e_{ij} \notin E \text{ with probability } 1 - p_{ij} \end{cases},$$

where p_i is the probability that the edge e_i appears in the graph. We will denote it as $E_i \sim Ber(p_i)$.

Edge Sets as Multivariate Bernoulli

The natural extension of this approach is to model any set of edges as a multivariate Bernoulli random variable $\mathbf{B} \sim Ber_k(\mathbf{p})$. $\mathbf{B}$ is uniquely identified by the parameter set

$$\mathbf{p} = \{p_I : I \subseteq \{1, \ldots, k\}, i \neq \varnothing\}, \qquad k = \frac{|\mathbf{V}|(|\mathbf{V}| - 1)}{2}$$

which represents the dependence structure [9] among the marginal distributions $B_i \sim Ber(p_i)$, $i = 1, \ldots, k$ of the edges.

The parameter set $\mathbf{p}$ can be estimated using a large number m of bootstrap samples as in Friedman *et al.* [3] or Imoto *et al.* [5], or MCMC samples as in Friedman & Koller [4].

Arcs and Univariate Trinomial Random Variables

Each arc a_{ij} in $\mathcal{G} = (\mathbf{V}, A)$ has three possible states, and therefore it can be modelled as a Trinomial random variable A_{ij}:

$$a_{ij} \sim A_{ij} = \begin{cases} -1 & \text{if } a_{ij} = \overleftarrow{a_{ij}} = \{v_i \leftarrow v_j\} \\ 0 & \text{if } a_{ij} \notin A, \text{ denoted with } \mathring{a_{ij}} \text{ .} \\ 1 & \text{if } a_{ij} = \overrightarrow{a_{ij}} = \{v_i \rightarrow v_j\} \end{cases}$$

As before, the natural extension to model any set of arcs is to use a multivariate Trinomial random variable $\mathbf{T} \sim Tri_k(\mathbf{p})$.
However:

- the acyclicity constraint of Bayesian networks makes deriving exact results very difficult because it cannot be written in closed form;
- the score equivalence of most structure learning strategies makes inference on $Tri_k(\mathbf{p})$ tricky unless particular care is taken (i.e. both possible orientations of many arcs result in equivalent probability distributions, so the algorithms cannot choose between them).

Measures of Structure Variability

Second Order Properties of $Ber_k(\mathbf{p})$ and $Tri_k(\mathbf{p})$

All the elements of the covariance matrix Σ of an edge set $\mathcal{E}$ are bounded,

$$p_i \in [0,1] \Rightarrow \sigma_{ii} = p_i - p_i^2 \in \left[0, \frac{1}{4}\right] \Rightarrow \sigma_{ij} \in \left[0, \frac{1}{4}\right],$$

and similar bounds exist for the eigenvalues $\lambda_1, \ldots, \lambda_k$,

$$0 \leqslant \lambda_i \leqslant \frac{k}{4} \qquad \text{and} \qquad 0 \leqslant \sum_{i=1}^{k} \lambda_i \leqslant \frac{k}{4}.$$

These bounds define a closed convex set in $\mathbb{R}^k$,

$$\mathcal{L} = \left\{ \Delta^{k-1}(c) : c \in \left[0, \frac{k}{4}\right] \right\}$$

where $\Delta^{k-1}(c)$ is the non-standard $k-1$ simplex

$$\Delta^{k-1}(c) = \left\{ (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k : \sum_{i=1}^{k} \lambda_i = c, \lambda_i \geqslant 0 \right\}.$$

Similar results hold for arc sets, with $\sigma_{ii} \in [0,1]$ and $\lambda_i \in [0,k]$.

Minimum and Maximum Entropy

These results provide the foundation for characterising three cases corresponding to different configurations of the probability mass in $\mathrm{P}(\mathcal{G}(\mathcal{E}))$ and $\mathrm{P}(\mathcal{G}(\mathcal{E}) \mid \mathcal{D})$:

- minimum entropy: the probability mass is concentrated on a single graph structure. This is the best possible configuration for $\mathrm{P}(\mathcal{G}(\mathcal{E}) \mid \mathcal{D})$, because only one edge set E (or one arc set A) has a non-zero posterior probability.
- intermediate entropy: several graph structures have non-zero probabilities. This is the case for informative priors $\mathrm{P}(\mathcal{G}(\mathcal{E}))$ and for the posteriors $\mathrm{P}(\mathcal{G}(\mathcal{E}) \mid \mathcal{D})$ resulting from real-world data sets.
- maximum entropy: all graph structures have the same probability. This is the worst possible configuration for $\mathrm{P}(\mathcal{G}(\mathcal{E}) \mid \mathcal{D})$, because it corresponds to a non-informative prior. In other words, the data $\mathcal{D}$ do not provide any information useful in identifying a high-posterior graph $\mathcal{G}$.

Properties of the Multivariate Bernoulli

In the minimum entropy case, only one configuration of edges E has non-zero probability, which means that

$$p_{ij} = \begin{cases} 1 & \text{if } e_{ij} \in E \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \Sigma = \mathbf{O}$$

where $\mathbf{O}$ is the zero matrix.
The uniform distribution over $\mathbf{G}$ arising from the maximum entropy case has been studied extensively in random graph theory [1]; its two most relevant properties are that all edges e_{ij} are independent and have $p_{ij} = \frac{1}{2}$. As a result, $\Sigma = \frac{1}{4} I_k$; all edges display their maximum possible variability, which along with the fact that they are independent makes this distribution non-informative for $\mathcal{E}$ as well as $\mathcal{G}(\mathcal{E})$.

Properties of the Multivariate Trinomial

In the maximum entropy case we have that [10]

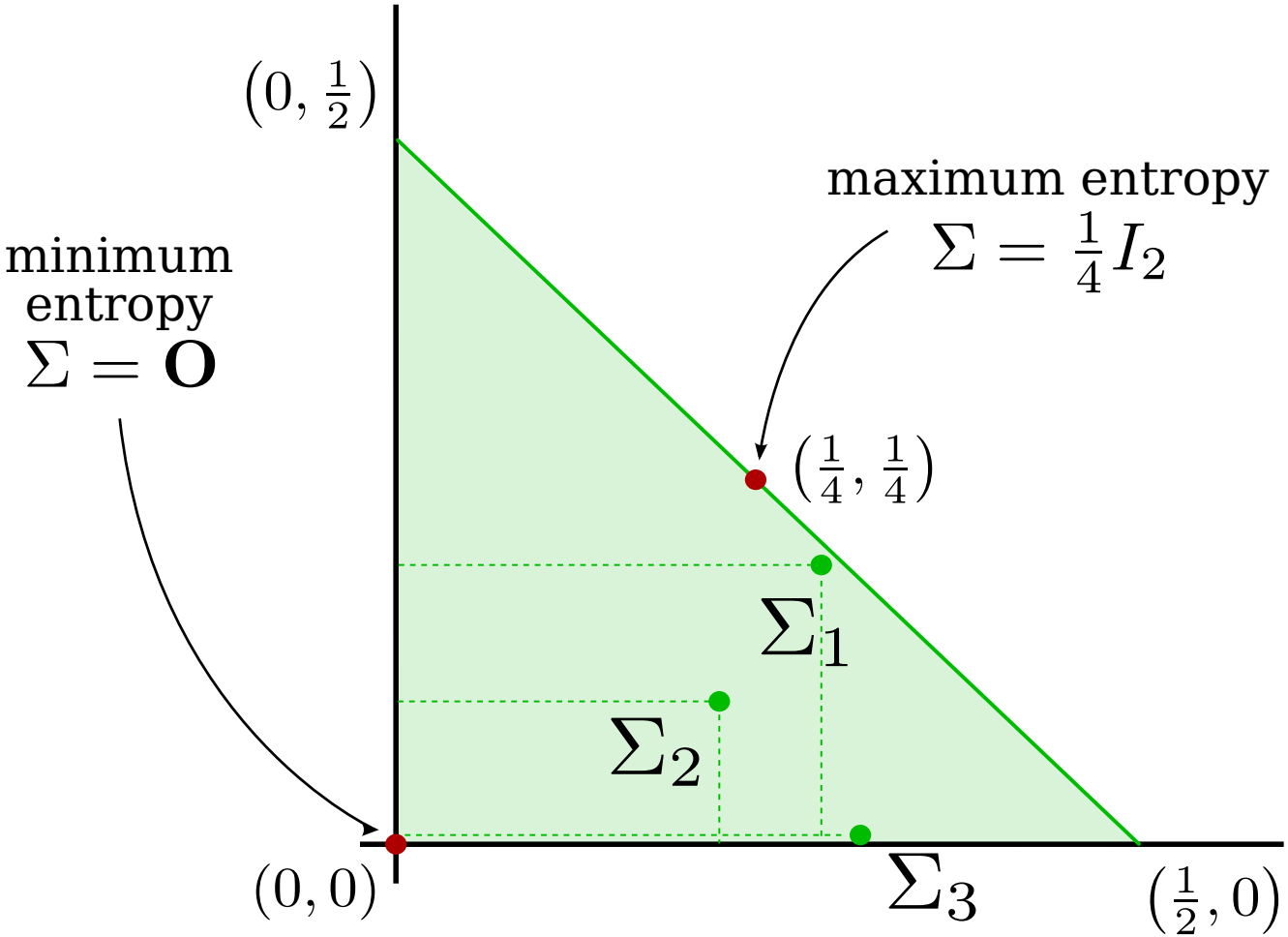
$$\mathrm{P}(\overrightarrow{a_{ij}}) = \mathrm{P}(\overleftarrow{a_{ij}}) \simeq \frac{1}{4} + \frac{1}{4(n-1)} \to \frac{1}{4}$$

$$\mathrm{P}(\mathring{a}_{ij}) \simeq \frac{1}{2} - \frac{1}{2(n-1)} \to \frac{1}{2}$$

as $n \to \infty$, where n is the number of nodes of the graph. As a result, we have that

$$\begin{aligned}
\mathrm{E}(A_{ij}) &= \mathrm{P}(\overrightarrow{a_{ij}}) - \mathrm{P}(\overleftarrow{a_{ij}}) = 0, \\
\mathsf{VAR}(A_{ij}) &= 2\,\mathrm{P}(\overrightarrow{a_{ij}}) \simeq \frac{1}{2} + \frac{1}{2(n-1)} \to \frac{1}{2}, \\
|\mathsf{COV}(A_{ij}, A_{kl})| &= 2\,[\mathrm{P}(\overrightarrow{a_{ij}}, \overrightarrow{a_{kl}}) - \mathrm{P}(\overrightarrow{a_{ij}}, \overleftarrow{a_{kl}})] \\
&\lessapprox 4 \left[\frac{3}{4} - \frac{1}{4(n-1)}\right]^2 \left[\frac{1}{4} + \frac{1}{4(n-1)}\right]^2 \to \frac{9}{64}.
\end{aligned}$$

A Geometric Representation of Entropy in $\mathcal{L}$



The space of the eigenvalues $\mathcal{L}$ for two edges in an undirected graph.

Univariate Measures of Variability

- The generalised variance, $\mathrm{VAR}_G(\Sigma) = \det(\Sigma) = \prod_{i=1}^{k} \lambda_i \in \left[0, \frac{1}{4^k}\right]$.
- The total variance (or total variability)

$$\mathrm{VAR}_T(\Sigma) = \mathrm{tr}\,(\Sigma) = \sum_{i=1}^{k} \lambda_i \in \left[0, \frac{k}{4}\right].$$

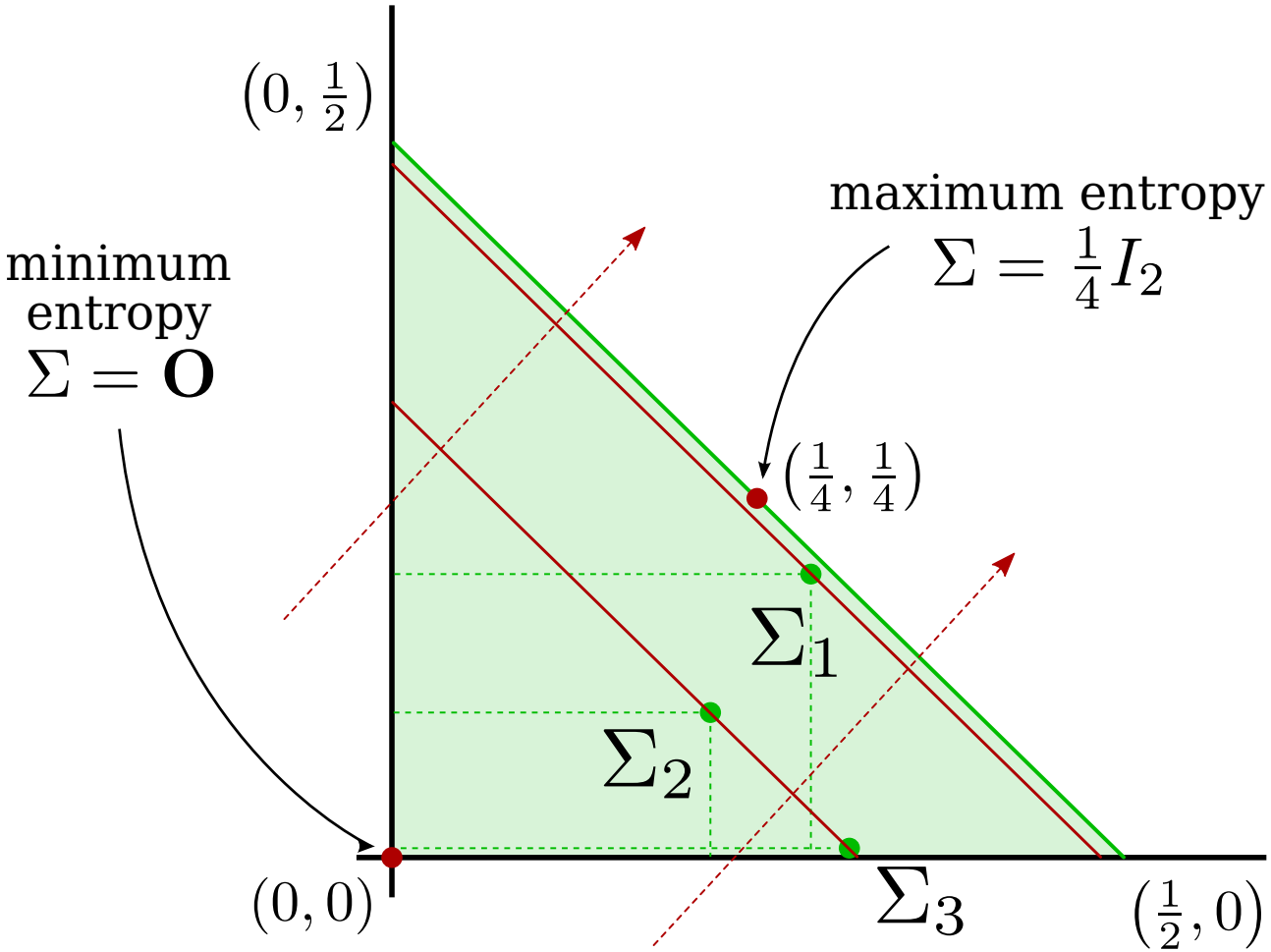
- The squared Frobenius matrix norm

$$\mathrm{VAR}_F(\Sigma) = |||\Sigma - \frac{k}{4} I_k|||_F^2 = \sum_{i=1}^{k} \left(\lambda_i - \frac{k}{4}\right)^2 \in \left[\frac{k(k-1)^2}{16}, \frac{k^3}{16}\right].$$

All of these measures can be rescaled to vary in the $[0, 1]$ interval and to associate high values to networks whose structure displays a high entropy.

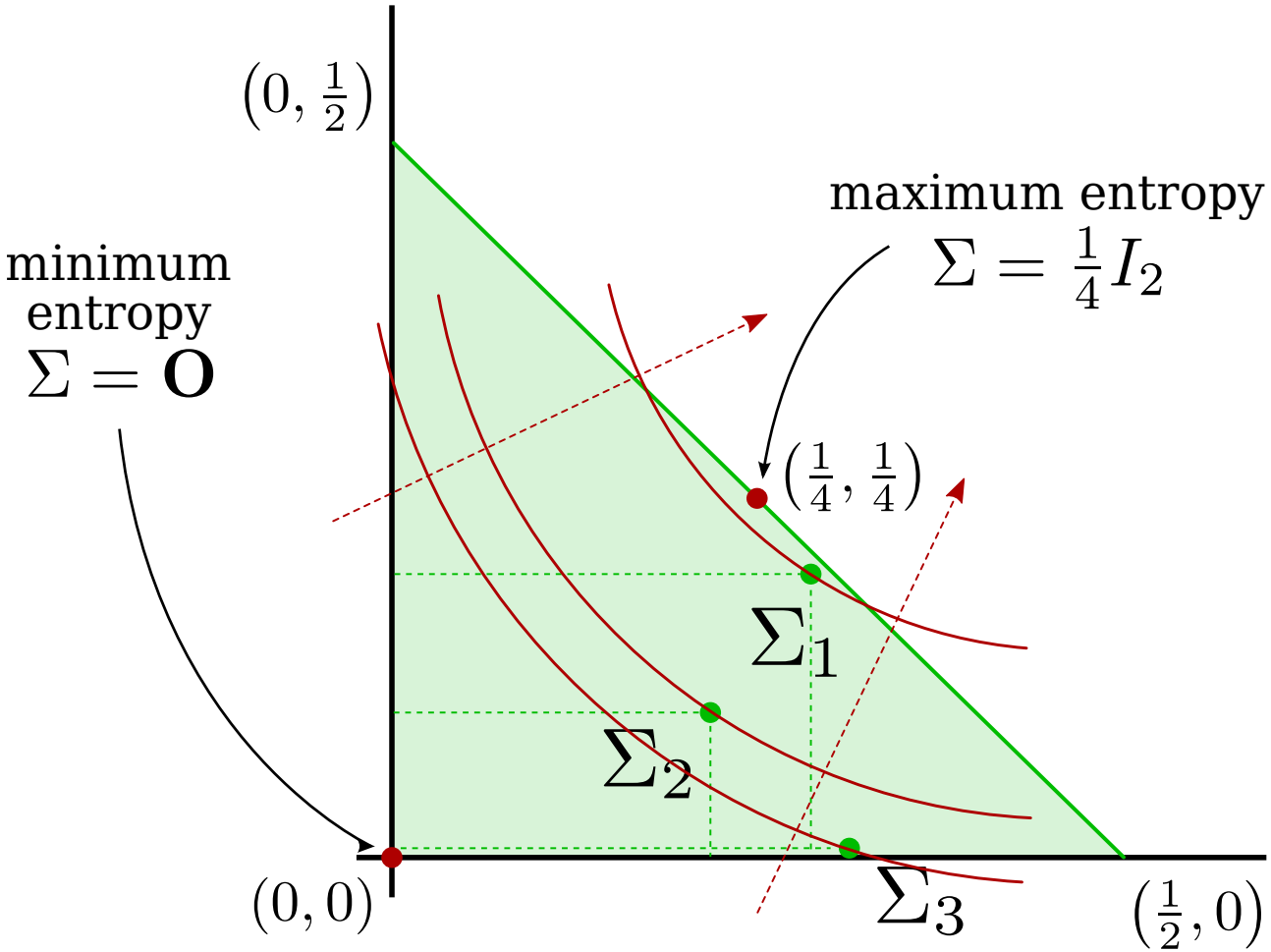
The equivalent measures of variability for directed acyclic graphs can be derived in the same way, and they can be similarly normalised.

Structure Variability (Total Variance)



Level curves in $\mathcal{L}$ for $\mathrm{VAR}_T(\Sigma)$.

Structure Variability (Squared Frobenius Matrix Norm)



Level curves in $\mathcal{L}$ for $\mathrm{VAR}_F(\Sigma)$.

Conclusions and Open Problems

- First and second order properties of $\mathrm{P}(\mathcal{G}(\mathcal{E}))$ and $\mathrm{P}(\mathcal{G}(\mathcal{E}) \mid \mathcal{D})$ can be often derived in closed form, and have a geometric interpretation.
- First and second order properties of the uniform $\mathrm{P}(\mathcal{G}(\mathcal{E}))$ on directed acyclic graphs can be a basis for simulations and the definition of new priors; could they translate to the uniform prior over decomposable undirected graphs?
- Is there a way of identifying paths using covariance matrix decompositions?
- Shrinking the covariance matrix affects $\mathrm{P}(e_{ij})$ and $\mathrm{P}(a_{ij})$ as well, and it is possible to use it for regularisation purposes. Applications to Bayesian model averaging and significant edges/arcs identification?

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